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PROPERTIES OF THE INVARIANT COMPONENTS OF THE WEYL TENSOR IMPLIED BY THE BIANCHI IDENTITIES†

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For vacuum regions in pseudo-Riemannian spaces in finite or infinitely large volumes of four-dimensional spaces, the Riemann tensor becomes the Weyl tensor. Based on the conditions of continuity for the space geometry, the form of the invariant components of the Weyl tensor, as functions of the coordinate arguments in the Fermi variables on canonical global time coordinate curves, is determined. With reference to a complete system of independent canonical invariants of the different types of Weyl tensor in four-dimensional spaces, the conditions of continuity are used to determine the form of these invariants as functions of the comoving individual Fermi coordinates at points of the pseudo-Riemannian spaces.

In order to construct various particular exact solutions in connection with the definition of different kinds of pseudo-Riemannian spaces that contain empty volumes V_4, the following partial differential tensor equations, which are direct corollaries of the Bianchi identities, must be satisfied in any system of coordinates inside V_4

$$\nabla^i(R_{ij} - {}^1/_2 g_{ij}R) = 0, \quad \text{or} \quad R_{ij} - {}^1/_2 g_{ij}R = T_{ij} = 0 \tag{1}$$

Hence $R = 0$ and $R_{ij} = 0$.

The definition of a vacuum requires that $T_{ij} = 0$ in finite or infinite volumes V_4 of the Riemannian spaces. Though all of the ensuing theory was developed directly for a vacuum, for the Weyl tensor, it can easily be extended, to the case when $T_{ij} = \kappa g_{ij}$, where κ is a constant, and the Gaussian curvature of the space $R = -4\kappa$ is constant. When $\kappa = 0$ the Riemann tensor is equal to the Weyl tensor, with $R_{ijkl} = W_{ijkl}$ and $W_{i.ks}^{s..} = 0$.

The construction proposed here to determine all possible solutions of system (1) is based on individualizing the points of the space, which are defined using a comoving system of coordinates ξ^α, τ; for such a system the metric may always be defined, without loss of generality, as follows:

$$ds^2 = c^2 d\tau^2 + 2g_{\alpha 4}(\xi^\alpha, \tau) d\xi^\alpha d\tau + g_{\alpha\beta}(\xi^\alpha, \tau) d\xi^\alpha d\xi^\beta \tag{2}$$

where the functions $g_{ij}(\xi^\alpha, \tau)$ are the components of the metric tensor which, in the case of the particular solutions sought here, are not defined uniquely but only up to a coordinate transformation.

A metric of the form (2) may be associated with a family of world lines L of individual points, defined by $\xi^\alpha = \text{const}$; the variable coordinate τ is then a global proper time on each

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line in L. On these lines $ds^2=c^2d\tau^2$, the four-dimensional velocity is $\bar{u}=d\bar{s}/d\tau$, and the absolute acceleration at each point of the world lines L is determined relative to the inertial tetrads with constant bases $e^i=\mathrm{const}$ by formulae of the type

$$\bar{a}=\frac{d\bar{u}}{d\tau}=\frac{c\partial g_{\alpha 4}(\xi^\gamma,\tau)}{\partial\tau}e^\alpha=\left(\frac{du^k}{d\tau}+u^p\Gamma^k_{4p}\right)e_k$$

with $e^i=\text{э}^i$, where the bases э^i are the variable contravariant basis vectors in comoving coordinates at each point of the family of lines L.

We introduced the concept of the global time in Riemannian spaces in [1, 2], where it was defined as a time coordinate τ in a comoving frame of reference; this is a direct generalization and analogue of the notion of absolute universal time in Newtonian continuum mechanics.

The subsequent steps of the construction, aimed at obtaining solutions of the system of equations (1), involve working with variable tetrads э^i at all points of the lines L and introducing further inertial tetrads S with constant bases e^i. Generally speaking, these constant bases will either be identical with э^i at a fixed instant of time, or their elements will be arbitrary vectors constant at all points of the space; in particular, they may be constant orthonormal bases, identical at fixed instants of time with variable and also with orthonormal canonical bases. In the general case the bases э^i and e^i are related by a transformation $x^\alpha=x^\alpha(\xi^1,\ \xi^2,\ \xi^3,\ \tau)$, where x^α, τ are the coordinates in the tetrads S.

It will be shown below that solutions of Eqs (1) exist in which the family of world lines is L, and a corresponding system of comoving coordinates, in which the individual points are named by equalities $\xi^\alpha=\mathrm{const}(\alpha)$. The curves L may be fixed fairly arbitrarily. In addition, however, in order to single out particular solutions one needs further data, which are formulated in terms of characteristics of the invariant parameters.

We shall now proceed to determine such parameters and ascertain their properties, which are derived from the components of the corresponding Riemann tensors.

In 1949 Petrov (see, for example, [3]) proposed to obtain fundamental solutions of Eqs (1) by introducing six-dimensional symmetric matrices K constructed from the real components of the Riemann tensor in four-dimensional space–time in appropriate frames of reference

$$K=\|K_{ab}\|=\begin{array}{c||ccc|ccc||l} & 14 & 24 & 34 & 23 & 31 & 12 & kl(a)\\ \hline 14 & \cdots & K_{1424} & \cdots & & & & \\ 24 & & & & & N & & \\ 34 & & M & & & & & \\ 23 & & & & \cdots & K_{2331} & \cdots & \\ 31 & & N & & & & & \\ 12 & & & & & M_1=-M & & \\ \hline ij(b) & & & & & & & \end{array} \tag{3}$$

where M and N for solutions of Eqs (1) are three-dimensional symmetric matrices of three possible types. In the canonical orthonormal tetrads for type T_1 we have

$$M=\left\|\begin{matrix}\alpha_1 & 0 & 0\\ 0 & \alpha_2 & 0\\ 0 & 0 & \alpha_3\end{matrix}\right\|,\quad N=\left\|\begin{matrix}\beta_1 & 0 & 0\\ 0 & \beta_2 & 0\\ 0 & 0 & \beta_3\end{matrix}\right\| \tag{4}$$

where $\lambda_s=-(\alpha_s+i\beta_s)$ are characteristic invariants—the roots of the corresponding "secular equation" for K. The roots λ_s for given Weyl tensors may differ at different points of the spaces.

It has been shown that for all three types

$$\alpha_1 + \alpha_2 + \alpha_3 = -\kappa, \quad \beta_1 + \beta_2 + \beta_3 = 0 \tag{5}$$

For type T_2, which corresponds to the Weyl tensor, we have double roots: $\lambda_2 = \lambda_3$ and $\kappa = 0$; in the canonical tetrads, along with the possible formulae (4), we may also have additional canonical formulae

$$M = \begin{Vmatrix} -2\alpha & 0 & 0 \\ 0 & \alpha + p & 0 \\ 0 & 0 & \alpha - p \end{Vmatrix}, \quad N = \begin{Vmatrix} -2\beta & 0 & 0 \\ 0 & \beta & p \\ 0 & p & \beta \end{Vmatrix} \tag{6}$$

For type T_3, with triple roots $\lambda_1 = \lambda_2 = \lambda_3 = -\kappa/3$ and $\beta_s = 0$, there may also be, besides solutions with canonical matrices of types (4) and (6), solutions with canonical matrices

$$M = \begin{Vmatrix} -\kappa/3 & p & 0 \\ p & -\kappa/3 & 0 \\ 0 & 0 & -\kappa/3 \end{Vmatrix}, \quad N = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & p & 0 \end{Vmatrix} \tag{7}$$

where p denotes arbitrary non-zero constants. For any family of world lines L, the real numbers α_s, β_s and κ corresponding to some solution of the equations in the canonical orthonormal tetrads S at isolated points may be arbitrary, provided they satisfy (5).

If the Weyl tensors are known, the three-dimensional part of the orientation of the canonical tetrads S on the world lines L may be determined algebraically from the matrices M and N in the six-dimensional notation for the matrix K.

In the inverse problem—the construction of fields of Weyl tensors—canonical inertial tetrads S with bases e^i along the world lines L may be designated by introducing an additional element of freedom in the Weyl tensor to be determined, namely, the choice of the three-dimensional orientation of the bases $э^\alpha$ at the points of the lines L.

The components W_{ijkl} of the Weyl tensor for a family of arbitrarily chosen world lines L, in an appropriate globally defined comoving canonical frame of reference (2) with variables ξ^α, τ, may be calculated in the space in tetrads S^* with bases $э^i$ and in the transformed inertial canonical bases e^i in the tetrads S.

At each point of four-dimensional pseudo-Riemannian space, in the local comoving tetrads with basis vectors $э^i$ and in the inertial tetrads e^i, we can write $d\xi^i э_i = dx^i e$. If $э_i = e_i$, then $d\xi^i = dx^i$, but the bases $э_i$ are variable both along a world line L and on making an infinitesimal transition to the points of a neighbouring world line L', while the inertial bases e_i may be introduced locally at each point, by definition, as constant bases in the tetrads S, in agreement with the equalities

$$dэ_i / d\xi^k = -\Gamma^s_{ik}(\xi^\gamma, \tau)э_s, \quad de_i / dx^k = 0 \tag{8}$$

An appropriate linear transformation with constant coefficients at each point of the space enables us to view the bases e_i and e^i as the same orthonormal inertial tetrad, forming its "own" local non-holonomic frames of reference at all points of the volume V_4.

One can establish a correspondence, for the same coordinates, between points in curved pseudo-Riemannian spaces and in Minkowski space (when there is topological equivalence) with the same Cartesian bases e^i in the tetrads S but, of course, with different metrics.

In the general case one can write the following equalities for Riemann tensors, and also for the components of the Weyl tensor, at each point of the space in the tetrad bases $э^i$ or e^i

$$W^*_{ijkl}\mathrm{Э}^i\mathrm{Э}^j\mathrm{Э}^k\mathrm{Э}^l = W_{ijkl}e^ie^je^ke^l \tag{9}$$

The Bianchi identities, which express the conditions of continuity implied by the definitions of the Riemann tensors, may be written as follows in each of the tetrads $\mathrm{Э}^i$ and e^i

$$\nabla_s W^*_{ijkl} + \nabla_k W^*_{ijls} + \nabla_l W^*_{ijsk} = W_{ijkl,s} + W_{ijls,k} + W_{ijsk,l} = 0 \tag{10}$$

where ∇_s, ∇_k, ∇_l are the symbols of the covariant derivatives in any systems of coordinates, and the subscripts set apart by commas denote partial derivatives with respect to the coordinates indexed s, k and l in the tetrads S.

Henceforth we will adhere to the assumption that $\mathrm{Э}^i = e^i$ and that $W^*_{ijkl} = W_{ijkl}$, but the partial derivatives of these components with respect to the coordinates are different. At the same time, Eq. (10) is meaningful even when $\mathrm{Э}^i \neq e^i$ and $W^*_{ijkl} \neq W_{ijkl}$.

If the Bianchi identities are satisfied in the local inertial tetrad systems, then it is obvious from (10) that they will be satisfied in any basis—in particular, in the global variables ξ^α, τ and $\mathrm{Э}^i$ at the points of the space.

We will now consider extensives, made up of the above components W^*_{ijkl} and W_{ijkl} at the points of space, in tetrad coordinates ξ^α, τ, $\mathrm{Э}^i$ and in coordinates x^α, τ, e^i for the inertial tetrads S on L; these extensives will satisfy all the conditions for algebraic symmetry of the components of the Riemann and Weyl tensors, respectively.

If the extensive of continuous functions W_{ijkl} additionally satisfies all the equalities (10), which are the tetrad Bianchi relations, then the totality of W^*_{ijkl} and W_{ijkl} can obviously be treated as the components in Lagrange coordinates ξ^i or in coordinates x^i for the Weyl tensor in local coordinates at points x^i for the inertial bases in the tetrads S.

As an extensive W_{ijkl} at each of the points ξ^i for the curvature tensor and Weyl tensor being constructed, one can take the canonical elements of the matrix K, expressed in terms of the canonical elements of the three-dimensional matrices M and N, which in turn depend on types T_1, T_2 and T_3 and are represented by formulae (4)–(6) in terms of the invariants α_s and β_s in the tetrads S.

For each given type, K is expressed in terms of M and N, hence also in terms of α_s and β_s, in the same way at all points of the space; but α_s and β_s, as functions of the Lagrange coordinates and in the corresponding tetrads S in variables x^i must satisfy not only (4)–(7), but also the Bianchi identities (10).

We will now proceed to establish additional formulae, valid at each point of the space, which require the invariants α_s and β_s as functions of x^i to satisfy the Bianchi conditions. Both the components W^*_{ijkl} in the global coordinates ξ^α, τ and the corresponding components W_{ijkl} in the local inertial tetrads must satisfy the Bianchi relations, taking into account transformations from x^α, τ variables to ξ^α, τ variables.

In the local orthonormal tetrads, comoving and canonical $\mathrm{Э}^i$, we also introduce inertial tetrads S with constant basis vectors e^i equal to $\mathrm{Э}^i$.

According to formulae (10) in the tetrads e^i, differentiation with respect to x^s in the tetrads S gives the Bianchi relations in the following form

$$\begin{aligned}
s=1,\quad & W_{1414,1} + W_{1441,1} + W_{1411,4} = 0\\
s=2,\quad & W_{1414,2} + W_{1442,1} + W_{1421,4} = 0\\
s=3,\quad & W_{1414,3} + W_{1443,1} + W_{1431,4} = 0\\
s=4,\quad & W_{1414,4} + W_{1444,1} + W_{1441,4} = 0
\end{aligned} \tag{11}$$

When the derivatives of the extensives W_{ijkl} in the bases e^i are determined, working in canonical tetrads in (10), one can use the canonical formulae for the components W_{ijkl}, expressed in terms of α_s and β_s as functions of x^i.

In the relations (11) for the components of the Weyl tensor in canonical form, allowance is made for the fact that all the non-diagonal terms in M and N in the extensive W_{ijkl} at neighbouring points vanish (in T_1) or are constants (in T_2 and T_3). In the second and third equations of (11), therefore, allowance must be made for the fact that, at all points of the space and in all types

$$W_{1412,1} = W_{1421,4} = W_{1443,1} = W_{1431,4} = 0 \tag{12}$$

The first and fourth equations of (11), in their exact form, must hold identically; we may therefore write

$$W_{1414} = \alpha_1(\tau, x^1)$$

In other words, it follows from (11) that the component W_{1414} may be any desired function — but of the variables τ and x^1 only.

Similarly, differentiation of the other diagonal terms W_{2424} and W_{3434} yields the following equalities in the canonical tetrads S

$$W_{1414} = \alpha_1(\tau, x^1), \quad W_{2424} = \alpha_2(\tau, x^2), \quad W_{3434} = \alpha_3(\tau, x^3)$$

Now, using (5), we obtain a formula for the Weyl tensor

$$\alpha_1(\tau, x^1) + \alpha_2(\tau, x^2) + \alpha_3(\tau, x^3) = 0 \tag{13}$$

It follows from this formula that on all the coordinate lines x^1, when $\tau = \text{const}$, $x^2 = \text{const}$, $x^3 = \text{const}$, the only possible variable quantity is $\alpha_1(x^1)$, but by (11) α_1 must be a constant, hence it is independent of x^1.

Similar conclusions follow from (13): α_2 and α_3 must be independent of x^2 and x^3. Consequently, the three invariants α_1, α_2 and α_3 for the given family of world lines may depend only on the specific value of the appropriate global time τ and the specific Weyl tensor under consideration, and moreover

$$\alpha_1(\tau) + \alpha_2(\tau) + \alpha_3(\tau) = 0 \tag{14}$$

We will now evaluate the canonical quantities β_1, β_2, β_3. We have

$$\begin{aligned} s = 1: &\quad W_{1423,1} + W_{1431,2} + W_{1412,3} = 0 \\ s = 2: &\quad W_{1423,2} + W_{1432,2} + W_{1422,3} = 0 \\ s = 3: &\quad W_{1423,3} + W_{1433,2} + W_{1432,3} = 0 \\ s = 4: &\quad W_{1423,4} + W_{1434,2} + W_{1442,3} = 0 \end{aligned} \tag{15}$$

Hence, since

$$W_{1431,2} = W_{1412,3} = W_{1434,2} = W_{1442,3} = 0$$

we conclude, using equations analogous to (15) for W_{2431} and W_{3412}, that

$$W_{1423} = \beta_1(x^2, x^3), \quad W_{2431} = \beta_2(x^3, x^1), \quad W_{3412} = \beta_3(x^1, x^2) \tag{16}$$

By (5) and (7), the Weyl tensor in the inertial tetrad S must also satisfy the equality

$$\beta_1(x^2, x^3) + \beta_2(x^3, x^1) + \beta_3(x^1, x^2) = 0 \tag{17}$$

For Weyl tensors of type D, with metrics corresponding to equal roots $\lambda_2 = \lambda_3$, it must also be true that

$\beta_2 = \beta_3$. In such cases, it follows immediately from (17) in the variables x^1, x^2, x^3 that $\beta_1 = -2\beta_2 = \text{const}$. Hence one easily concludes that for different Kerr solutions with fixed families of world lines L, namely, circles with their centres on the axis of symmetry, the invariants β_s may be different constants on the same orbital circle, with similar formulae for the metrics in the comoving Lagrange systems.

Formula (17) enables further simplifications to be made. Differentiating (17) with respect to x^1 we obtain

$$\partial\beta_2(x^1, x^3)/\partial x^1 + \partial\beta_3(x^2, x^1)/\partial x^1 = 0$$

Hence, differentiating with respect to x^3, we obtain

$$\partial^2\beta_2(x^1, x^3)/\partial x^1\partial x^3 = 0$$

and therefore $\beta_2 = \psi(x^1) + \varphi(x^3)$. Similarly one obtains $\beta_3 = \psi(x^2) + f(x^1)$ and $\beta_1 = g(x^2) + h(x^3)$.
By (17) β_1, β_2 and β_3 may be converted to the form

$$\begin{aligned} \beta_1 &= f(x^2) - h(x^3) + \beta_{10} \\ \beta_2 &= h(x^3) - \varphi(x^1) + \beta_{20} \\ \beta_3 &= \varphi(x^1) - f(x^2) + \beta_{30} \end{aligned} \tag{18}$$

where β_{s0} are scalar constants on the world lines of L, equal to the values of $\beta_s(\xi^\alpha, \tau)$ corresponding at the relevant point of space C to the coordinates of the centre in the inertial tetrad.

Thus, the solutions to problems involving the determination of different Weyl tensors may be expressed in terms of functions that can be selected arbitrarily for a family of comoving world lines L with three-dimensional canonical tetrads e^α of scalars $\lambda_s(\xi^\alpha, \tau) = -(\alpha_s + i\beta_s)$ and, in addition, with fixed particular examples of Petrov's matrices K for types T_1, T_2 and T_3. (The corresponding Weyl tensor depends on the system of canonical inertial tetrads e^i at points on the world lines of the family L.)

The rest of the theory consists in computing global functions for the components of the metric $g_{ij}(\xi^\alpha, \tau)$ in comoving coordinates, containing additional arbitrary elements because of possible coordinate transformations.

The following result is obvious in the context of the theory presently under discussion. If all the invariants α_s and β_s are given as arbitrary constants, independently of the coordinates of the points of the space, except that they satisfy (5), then one obtains the Weyl tensor in the appropriate comoving Lagrange system of coordinates ξ^α, τ.

In view of our previous conclusions, it is obvious that in this situation, in type T_2, i.e. in the case of double roots, such as $\alpha_2 = \alpha_3$ and $\beta_2 = \beta_3$ when $p = \text{const}$, conditions (5) and (14) take on a simpler but similar form, while in type T_3 the Bianchi relations become identities, since all the components of the Riemann tensor in the canonical matrix K are constant.

In type N, when $\lambda_1 = \lambda_2 = \lambda_3 = 0$ with $p = 1$, formulae (6) are true with $\alpha_s = 0$ at each point of the space N. Therefore the corresponding components of the Weyl tensor in the canonical matrices will always satisfy the Bianchi identities. Consequently, one obtains a corresponding formula (6) for the components of the Weyl tensor in canonical systems.

Hence, the fundamental problems in constructing Riemannian spaces in empty volumes involve the investigation of Weyl tensors in comoving coordinates as carriers, in the sense of the general theory of relativity, of the characteristic properties of gravitational phenomena in vacuum regions. In the general case, however, the resulting set of all possible metrics such that $R_{ij} = 0$ or $R_{ij} = -\kappa g_{ij}$, and the corresponding spaces, are suitable for a gravitational theory involving additional conditions and restrictions.

If the problem of determining the components of the metric tensor g_{ij} and the Weyl tensor W_{ijkl} in the appropriate coordinates has been solved, then the solution as viewed by any other given observer in the space may be obtained by carrying out a coordinate transformation from the computed solution in the comoving frame to the observer's frame, using an inertial navigational algorithm. An account of the computational methods of inertial navigation may be found in [4].

The problem of choosing a particular frame of reference is as a rule not trivial in theories that work with canonical and local non-holonomic inertial frames; this point is of fundamental importance in applications.

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